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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before September 3, 2012

Problems

43. *Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National Colege, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Jointly).* Let a be a positive real number and $\Gamma(x)$ be the Gamma function (or Euler's second integral). Calculate

$$\lim_{x\to\infty}\left((x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)-x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)\right)$$

44. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let A denote the Glaisher–Kinkelin constant defined by

$$A=\lim_{n\to\infty}n^{-n^2/2-n/2-1/12}e^{n^2/4}\prod_{k=1}^{n}k^k=1.282427130\ldots.$$

Prove that

$$\sum_{p=1}^{\infty}\frac{\zeta(2p+1)-1}{p+2}=-\frac{\gamma}{2}-6\ln A+\ln 2+\frac{7}{6},$$

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where ζ is the Riemann zeta function defined by $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ for $\Re(s) > 1$.

45. *Proposed by Moubinool Omarjee, Paris, France.* Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$. Prove that there exist $(A, B, C) \in M_2(\mathbb{Z}) \times M_2(\mathbb{Z}) \times M_2(\mathbb{Z})$ such that $M = A^2 + B^2 + C^2$.

46. *Proposed by Anastasios Kotronis, Athens, Greece, and Serafeim Tsipelis, Ionnina, Greece (Jointly).* Evaluate

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{\zeta(2n)}{n},$$

where ζ is the Riemann zeta function.

47. *Proposed by David R. Stone, Georgia Southern University, Statesboro, GA, USA.* Identify the graph of the equation

$$y^4 - 10x^2y^3 + 25x^4y^2 - 50x^6y + 24x^8 = 0$$

48. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Let $f : (0, \infty) \to (0, \infty)$ be a differentiable function for which there exist real numbers a and b, $0 < a < b$ such that

(i): $\int_a^b \frac{f(x)}{x^2} dx = \frac{f(b)}{b} - \frac{f(a)}{a}$

(ii): $\frac{f'(a)}{f(a)} = \frac{2}{a}$

(a) Find an example of a function that satisfies the preceding conditions

(b) Show that there exists a $c \in (a, b)$ such that

$$\frac{f(c)}{f(a)} = \left(\frac{c}{a}\right)^2 e^{(c-a)\left(\frac{f'(c)}{f(c)} - \frac{2}{c}\right)}$$

49. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{\sqrt[3]{(a-b)^2(b-c)^2(c-a)^2}}{ab + bc + ca}$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

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36. *Proposed by Anastasios Kotronis, Athens, Greece.* Evaluate the sum

$$\sum_{n=1}^{+\infty} n\left(2^{-1/2}-1+\binom{1/2}{1}\frac{1}{2}-\binom{1/2}{2}\frac{1}{4}+\cdots+(-1)^{n+1}\binom{1/2}{n}\frac{1}{2^n}\right)$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The value of the sum to be computed is obtained immediately applying the following

Lemma. *Suppose that f has a power series expansion $\sum_{k=0}^{\infty} a_k x^k$ that converges in $(-\rho,\rho)$. For a given $x\in(-\rho,\rho)$ and $n\geq 0$ we define the rest $R_n(x)$ by*

$$R_n(x)=f(x)-\sum_{k=0}^{n} a_k x^k=\sum_{k=n+1}^{\infty} a_k x^k$$

Then for every $x\in(-\rho,\rho)$, we have

$$\sum_{n=1}^{\infty} nR_n(x)=\frac{x^2}{2}f''(x)$$

Proof. For $x\in(-\rho,\rho)$ let r be a positive real number such that $|x|<r<\rho$ and let $M_r=\sup\{|a_k|r^k : k\geq 0\}$. Then

$$|R_n(x)|\leq\sum_{k=n+1}^{\infty}|a_k|r^k\left(\frac{|x|}{r}\right)^k\leq\frac{M_r|x|}{r-|x|}\left(\frac{|x|}{r}\right)^n$$

This implies the absolute convergence of the following two series :

$$S_1=\sum_{n=1}^{\infty}\frac{n(n+1)}{2}R_n(x)\quad\text{and}\quad S_2=\sum_{n=1}^{\infty}\frac{n(n-1)}{2}R_n(x)$$

On one hand we have

$$S_1-S_2=\sum_{n=1}^{\infty}\left(\frac{n(n+1)}{2}-\frac{n(n-1)}{2}\right)R_n(x)=\sum_{n=1}^{\infty}nR_n(x)$$

and on the other hand, since $R_{n-1}(x) = a_n x^n + R_n(x)$, we have

$$\begin{aligned} S_2 &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} R_n(x) \\ &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} R_{n-1}(x) - \sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n x^n \\ &= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} R_n(x) - \frac{x^2}{2} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= S_1 - \frac{x^2}{2} f''(x) \end{aligned}$$

Now the Lemma follows by combining the preceding results.

Applying the Lemma to the function $f(x) = \sqrt{1+x}$ and the point $x = -1/2$ gives

$$\sum_{n=1}^{\infty} n \left(2^{-1/2} - 1 + \binom{1/2}{1} \frac{1}{2} - \binom{1/2}{2} \frac{1}{4} + \cdots + (-1)^{n+1} \binom{1/2}{n} \frac{1}{2^n} \right) = -\frac{\sqrt{2}}{16}$$

and we are done.

Also solved by Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Albert Stadler, Switzerland; Tsipelis Serafeim, Ioannina, Greece; Kostas Tsouvalas, Lemnos, Greece; and the proposer

37. *Proposed by Mihály Bencze, Braşov, Romania.* If $A, B \in M_2(\mathbb{R})$ then prove that

$$2\,(\det A)^2 + \det(AB + BA) + 2\,(\det B)^2 \geq \det\left(A^2 - B^2\right) + 4 \det AB$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. First we will prove following

Lemma. *Let* $A, B \in M_2(\mathbb{R})$. *Then* $\det(A+B) + \det(A-B) = 2(\det A + \det B)$.

Proof. Consider the polynomial

$$P(x) = \det(A + Bx) = x^2 \det B + \alpha x + \det A,$$

then we have

$$P(1) + P(-1) = \det(A+B) + \det(A-B)$$

On the other hand

$$\begin{aligned} P(1) + P(-1) &= (\det B + \alpha + \det A) + (\det B - \alpha + \det A) \\ &= 2(\det A + \det B) \end{aligned}$$

From the preceding, we get

$$\det(A+B) + \det(A-B) = 2(\det A + \det B)$$

as desired.

Now we prove statement. Indeed, by the lemma we have

$$\det(A^2 - B^2) = 2(\det A^2 + \det B^2) - \det(A^2 + B^2)$$

So it is enough to prove following inequality

$$\det(A^2+B^2)+\det(AB+BA)\geq 4\,\det(AB)$$

Again, by the lemma we have

$$\begin{aligned}\det(A^2+B^2+AB+BA)&+\det(A^2+B^2-AB-BA)\\&=2(\det(A^2+B^2)+\det(AB+BA))\end{aligned}$$

On other hand,

$$\begin{aligned}\det(A^2+B^2+AB+BA)&+\det(A^2+B^2-AB-BA)\\&=\det(A+B)^2+\det(A-B)^2\\&=(\det(A+B))^2+(\det(A-B))^2\end{aligned}$$

From the preceding, we obtain

$$(\det(A+B))^2+(\det(A-B))^2\geq 8\det(AB)$$

By the wel–known inequalities $\forall x,y\in\mathbb{R}:\ x^2+y^2\geq\frac{(x+y)^2}{2},\ x^2+y^2\geq 2xy$ and the lemma, we get

$$\begin{aligned}(\det(A+B))^2+(\det(A-B))^2&\geq\frac{1}{2}(\det(A+B)+\det(A-B))^2\\&=\frac{1}{2}(2(\det A+\det B))^2\\&=2((\det A)^2+(\det B)^2+2\det A\det B)\\&\geq 2(2\det A\det B+2\det A\det B)\\&=8\det A\det B=2\det(AB)\end{aligned}$$

and we are done.

Also solved by Moubinool Omarjee, Paris, France; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Albert Stadler, Switzerland; Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania; and the proposer.

38. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Determine all functions $f:[0,1]\to\mathbb{R}$ that have the following properties

a): f is three times differentiable with $f'''(x)\geq 0,\forall x\in[0,1]$;
b): f' is increasing and strictly positive;
c): $f'(1)\Big(2\left(f(1)-f(0)\right)-f'(1)\Big)\int_0^1\frac{dx}{(f'(x))^2}=1$

Solution by the proposer. From $\left(\frac{x}{f'(x)}\right)'=\frac{f'(x)-xf''(x)}{(f'(x))^2}dx$, $\forall x\in[0,1]$, we have

$$\int_0^1\left(\frac{x}{f'(x)}\right)'dx=\int_0^1\frac{f'(x)-xf''(x)}{(f'(x))^2}dx\quad\text{and}\quad\frac{1}{f'(1)}=\int_0^1\frac{f'(x)-xf''(x)}{(f'(x))^2}dx$$

Consider the function $g:[0,1]\to\mathbb{R}$, defined by $g(x)=f'(x)-xf''(x)$. Since $g'(x)=-xf'''(x)\leq 0$, then g is decreasing. Applying Cebyshev's inequality to the

decreasing functions $g(x)$ and $\frac{1}{(f'(x))^2}$, we have

$$\begin{aligned}\frac{1}{f'(1)} &= \int_0^1 \frac{f'(x) - xf''(x)}{(f'(x))^2} dx \\ &\geq \int_0^1 \frac{dx}{(f'(x))^2} \cdot \int_0^1 (f'(x) - xf''(x))\, dx \\ &= \int_0^1 \frac{dx}{(f'(x))^2} \left(f(1) - f(0) - f'(1) + \int_0^1 f'(x) dx \right) \\ &= \int_0^1 \frac{dx}{(f'(x))^2} \left(2\left(f(1) - f(0)\right) - f'(1) \right)\end{aligned}$$

From the preceding and the condition in **c**) it follows that $\frac{1}{(f'(x))^2}$ or g are constant functions.

1. If $f'(x) = a, \forall x \in [0,1]$ then$f(x) = ax + b$ with $a, b \in \mathbb{R}$ and $a > 0$.
2. If $g(t) = a$, then $f'(t) - tf'(t) = a, \forall t \in [0,1]$. Thus,

$$\int_0^x (f'(t) - tf''(t))\, dt = cx, \forall x \in [0,1] \Rightarrow f(x) - f(0) - xf'(x) + \int_0^x f'(t) dt = cx$$

and

$$f'(x) = \frac{2}{x} f(x) + \left(-\frac{f(0)}{x} - c \right), \forall x \in (0,1]$$

We observe that the last expression is a first order linear differential equation of the form: $y' = p(x)y + q(x)$, with solutions $y = e^{P(x)} \int q(x) e^{-P(x)} dx$, where $P(x)$ is a fixed primitive function. Since $P(x) = 2\ln x$, then we have

$$\begin{aligned}f(x) =& e^{2\ln x} \int \left(-\frac{f(0)}{x} - c \right) e^{-2\ln x} dx \\ =& x^2 \int \left(-\frac{f(0)}{x} - c \right) \frac{1}{x^2} dx \\ =& x^2 \left(\frac{f(0)}{2x^2} + \frac{c}{x} + d \right) \\ =& dx^2 + cx + \frac{f(0)}{2}, \forall x \in (0,1]\end{aligned}$$

Since f is continuous on $[0,1]$, then $f(x) = dx^2 + cx + \frac{f(0)}{2}, \forall x \in [0,1]$.
Taking into account the above and observing the conditions given in the statement, we obtain

$$f(x) = ax^2 + bx + c, \text{ with } a \geq 0, b > 0$$

39. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* If A, B and C are the angles of a triangle, then prove that:

$$\frac{\sin^k A + \sin^k B + \sin^k C}{A + B + C} \leq \frac{1}{3} \left(\frac{\sin^k A}{A} + \frac{\sin^k B}{B} + \frac{\sin^k C}{C} \right),$$

where $k \in (0,1]$.

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. WLOG we can assume $A \geq B \geq C$. If

$$\frac{\sin^k A}{A} \leq \frac{\sin^k B}{B} \leq \frac{\sin^k C}{C}$$

then by Chebyshev's Inequality, we have

$$\sum_{cyclic} \sin^k A \leq \frac{1}{3} \sum_{cyclic} A \sum_{cyclic} \frac{\sin^k A}{A}$$

Now we need to prove that the function $f(x) = \frac{\sin^k x}{x}$ is decreasing. Indeed,

$$f'(x) = \frac{\sin^{k-1} x}{x^2} (kx \cos x - \sin x)$$

Consider function $g(x) = kx \cos x - \sin x, x \in [0, \pi)$. Then,

$$\begin{aligned} g'(x) &= k \cos x - kx \sin x - \cos x \\ &= (k-1) \cos x - kx \sin x \leq 0 \end{aligned}$$

We have $x \in [0, \pi) \Rightarrow g(x) \leq g(0) = 0$. From this result we can easily see that $f(x)$ is decreasing function.

Also solved by Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; and the proposer.

Editors Comment: Using similar techniques Adrian Naco proved a generalized version of the above problem. Namely, he showed that

If $0 < x_1 \leq x_2 \leq \ldots \leq x_n < \pi$, then

$$\frac{\sum_{k=1}^n \sin^k x_i}{\sum_{k=1}^n x_i} \leq \frac{1}{n} \sum_{k=1}^n \frac{\sin^k x_i}{x_i},$$

where $k \in (0, 1]$.

40. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Find all distinct positive integers x, y, z, t all greater than 2, such that

$$\frac{x^3}{x-1} + \frac{y^3}{y-1} + \frac{z^3}{z-1} + \frac{t^3}{t-1}$$

is an integer.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Our solution will consist of two steps: first we will solve an apparently simpler problem (the next lemma), then we will reduce the considered problem to the simpler one.

Lemma. *Let M denote the set of quadruples (a, b, c, d) of positive integers such that $1 < a < b < c < d$ and let*

$$S = \left\{ (a, b, c, d) \in M : \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1 \right\}$$

Then S consists exactly of the following six elements :

$$\begin{array}{ccc} (2,3,7,42), & (2,3,8,24), & (2,3,9,18), \\ (2,3,10,15), & (2,4,5,20), & (2,4,6,12). \end{array}$$

Proof. Consider $(a,b,c,d) \in S$. If $a \geq 3$ then we would have $b \geq 4$, $c \geq 5$ and $d \geq 6$. This would imply that

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{57}{60} < 1$$

which is impossible. So $a = 2$. Now, if $b \geq 6$ then we would have $c \geq 7$, $d \geq 8$ and this would imply that

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{2} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{157}{168} < 1$$

which is absurd. Therefore $b \in \{3,4,5\}$. So let us consider the following cases: (i) If $b = 5$ then $\frac{1}{c} + \frac{1}{d} = \frac{7}{10}$, but $6 \leq c < d$ implies that $20d > 10(c+d) = 7cd \geq 42d$, which is absurd. (ii) If $b = 4$ then $\frac{1}{c} + \frac{1}{d} = \frac{1}{4}$, or equivalently $(d-4)(c-4) = 16$. But $1 \leq c-4 < d-4$, so $c-4$ is a positive divisor of 16 which is smaller than 4. *i.e.* $c-4 = 1$ or $c-4 = 2$. This yields the following two solutions:

$$(2,4,5,20) \quad \text{and} \quad (2,4,6,12)$$

(iii) If $b = 3$ then $\frac{1}{c} + \frac{1}{d} = \frac{1}{6}$, or equivalently $(d-6)(c-6) = 36$. But $-2 \leq c-6 < d-6$, so $c-6$ is a positive divisor of 36 which is smaller than 6. *i.e.* $c-6 \in \{1,2,3,4\}$. This yields the following four solutions:

$$(2,3,7,42), \quad (2,3,8,24), \quad (2,3,9,18) \quad \text{and} \quad (2,3,10,15)$$

This concludes the proof of the Lemma.

Let us come to our problem. Note that the expression given in the statement is symmetric in x,y,z,t. So if $(x,y,z,t) = (u_1,u_2,u_3,u_4)$ is a solution to the problem then all 24 permutations $(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}, u_{\sigma(4)})$, where $\sigma \in S_4$, are also solutions. Among these solutions there is only one that satisfies $x < y < z < t$, such a solution will be called an *ordered solution*. Let us denote by N the set of quadruples of integers (x,y,z,t) that satisfy $3 \leq x < y < z < t$. According to the remark above, it is enough to find all the solutions to the problem that belong to N. Since

$$\frac{u^3}{u-1} = 1 + u + u^2 + \frac{1}{u-1},$$

we see that

$$\frac{x^3}{x-1} + \frac{y^3}{y-1} + \frac{z^3}{z-1} + \frac{t^3}{t-1}$$

is an integer for some $(x,y,z,t) \in N$, if and only if,

$$f(x,y,z,t) = \frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} + \frac{1}{t-1}$$

is an integer. But, for $(x,y,z,t) \in N$, we have

$$f(x,y,z,t) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} < 2$$

Therefore,

$$\begin{aligned}\big\{(x,y,z,t)\in N : f(x,y,z,t)\in\mathbb{N}\big\} &= \big\{(x,y,z,t)\in N : f(x,y,z,t)=1\big\}\\ &= \big\{(x,y,z,t)\in N : (x-1,y-1,z-1,t-1)\in S\big\}\end{aligned}$$

where S is identified in the lemma. So the ordered solutions to our problem are

$$\begin{array}{ccc}(3,4,8,43), & (3,4,9,25), & (3,4,10,19),\\ (3,4,11,16), & (3,5,6,21), & (3,5,7,13).\end{array}$$

and the total number of solutions is 144.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.

41. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Evaluate

$$\int_0^{\pi/2} 4(\cos x)^2\left(\ln(\cos x)\right)^2 dx$$

Solution by the proposer. Setting $x=\arctan t$ we get

$$\int_0^{+\infty}\frac{\left(\ln(1+t^2)\right)^2}{(1+t^2)^2}dt$$

We write

$$\begin{aligned}\frac{d}{da}\int_0^{+\infty}\frac{\left(\ln(1+at^2)\right)^2}{(1+t^2)^2}dt &= \int_0^{+\infty}\frac{2\ln(1+at^2)t^2}{(1+t^2)^2(1+at^2)}dt\\ &=2\sqrt{a}\int_0^{+\infty}\frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2}dt=\sqrt{a}\int_{-\infty}^{+\infty}\frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2}dt\end{aligned}$$

Standard theorem on the exchange between integral and derivatives allow us to differentiate under the integral. Considering the complex function

$$\frac{z^2\mathrm{Ln}(z^2+1)}{(1+z^2)(z^2+a)^2}=\frac{z^2\mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2}+\frac{z^2\mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2}$$

we may write the two integrals

$$\sqrt{a}\int_{-\infty}^{+\infty}\frac{z^2\mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2}+\sqrt{a}\int_{-\infty}^{+\infty}\frac{z^2\mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2},$$

where $\mathrm{Ln}(z)=\ln(|z|)+i\vartheta$.

Let us consider the complex function $f_1(z)=\frac{z^2\mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2}$ and cut the complex plane along the set $Rez=0$, $Imz=[-i,-\infty)$. We compute the counterclockwise integral over the two curves

$$\gamma_1(t)=\{z\in\mathbf{C}: z=t,\ -r\le t\le r\},$$
$$\gamma_2(t)=\{z\in\mathbf{C}: z=re^{it},\ 0\le t\le\pi\},$$

$$\int_{\gamma_1\cup\gamma_2}f_1(z)dz=2\pi i(Resf_1(i)+Resf_1(i\sqrt{a}))$$

Moreover by defining $f_2(z) = \frac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2}$ and cutting the complex plane along the set $Rez = 0$, $Imz = [i, +\infty)$, we calculate the clockwise integral over the two curves

$$\gamma_1(t) = \{z \in \mathbf{C} : z = t,\ -r \leq t \leq r\},$$
$$\gamma_2(t) = \{z \in \mathbf{C} : z = re^{-it},\ -2\pi \leq t \leq -\pi\},$$
$$\int_{\gamma_1 \cup \gamma_2} f_2(z)dz = -2\pi i(Resf_2(-i) + Resf_2(-i\sqrt{a}))$$

Now we compute the following residues:

$$2\pi i\,(Resf_1(i) + Resf_2(-i)) = \frac{-2\pi i \mathrm{Ln}(2i)}{2i(a-1)^2} - \frac{2\pi i \mathrm{Ln}(-2i)}{2i(a-1)^2} = \frac{-\pi 2 \ln 2}{(a-1)^2}$$

$$Resf_1(i\sqrt{a}) = \lim_{z \to i\sqrt{a}} \frac{d}{dz} \frac{z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^2} =$$
$$= \lim_{z \to i\sqrt{a}} \Big(\frac{z^2}{(z+i)(1+z^2)(z+i\sqrt{a})^2} + \frac{2z\mathrm{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^2} +$$
$$- \frac{2z^3 \mathrm{Ln}(z+i)}{(1+z^2)^2(z+i\sqrt{a})^2} - \frac{2z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^3} \Big) \doteq a_3 + a_4 + a_5 + a_6$$

$$Resf_2(-i\sqrt{a}) = \lim_{z \to -i\sqrt{a}} \frac{d}{dz} \frac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^2} =$$
$$= \lim_{z \to -i\sqrt{a}} \Big(\frac{z^2}{(z-i)(1+z^2)(z-i\sqrt{a})^2} + \frac{2z\mathrm{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^2} +$$
$$- \frac{2z^3 \mathrm{Ln}(z-i)}{(1+z^2)^2(z-i\sqrt{a})^2} - \frac{2z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^3} \Big) \doteq a_7 + a_8 + a_9 + a_{10}$$

$$a_3 - a_7 = \frac{1}{2i(1-a)((1+\sqrt{a})}$$
$$a_4 - a_8 = \frac{-i\ln(1+\sqrt{a})}{\sqrt{a}(1-a)}$$
$$a_5 - a_9 = \frac{-i\sqrt{a}\ln(1+\sqrt{a})}{(1-a)^2}$$
$$a_6 - a_{10} = \frac{i\ln(1+\sqrt{a})}{\sqrt{a}(1-a)^2}$$

and then

$$\sqrt{a}\int_{-\infty}^{+\infty} \frac{z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2} + \sqrt{a}\int_{-\infty}^{+\infty} \frac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2} =$$
$$\frac{-2\pi \ln 2\sqrt{a}}{(a-1)^2} + \frac{\pi\sqrt{a}}{(1-a)(1+\sqrt{a})} + \frac{\pi\ln(1+\sqrt{a})}{(1-a)} + \frac{2\pi a \ln(1+\sqrt{a})}{(1-a)^2}$$

Now we must integrate respect to the variable a (we assume $0 < a < 1$ but $a > 1$ would yield of course the same result) and we get

$$\sqrt{a}\int_{-\infty}^{+\infty}\frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2}dt =$$
$$-2\pi\ln 2\left(\frac{1}{2(1-\sqrt{a})}-\frac{1}{2(1+\sqrt{a})}+\frac{1}{2}\ln\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)+$$
$$\pi\left(-\frac{1}{1+\sqrt{a}}-\frac{3}{2}\ln(1+\sqrt{a})-\frac{1}{2}\ln(1-\sqrt{a})\right)+$$
$$\pi\left(-\frac{1}{2}\ln^2(1+\sqrt{a})-\ln 2\cdot\ln(1-\sqrt{a})+\mathrm{Li}_2\left(\frac{1-\sqrt{a}}{2}\right)\right)+$$
$$+2\pi\left(\frac{1}{4}\ln(1-\sqrt{a})+\frac{1}{4}\frac{(1+\sqrt{a})\ln(1+\sqrt{a})}{1-\sqrt{a}}+\ln 2\cdot\ln(1-\sqrt{a})-\mathrm{Li}_2\left(\frac{1-\sqrt{a}}{2}\right)+\right.$$
$$\left.+\frac{1}{2}\frac{\ln(1+\sqrt{a})}{1+\sqrt{a}}+\frac{1}{2}\frac{1}{1+\sqrt{a}}+\frac{1}{2}\ln^2(1+\sqrt{a})\right)+C$$

where as usual $\mathrm{Li}_2(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^2}=-\int_0^z\ln(1-w)\frac{dw}{w}$, $w\in\mathbf{C}\backslash\{w\in\mathbf{C}\colon Imw=0, Rew\in[1,+\infty)\}$. The integrations are standard so we explain only the third integral

$$\int\frac{\ln(1+\sqrt{a})}{a-1}da\underbrace{=}_{a=y^2}\int\left(\frac{\ln(1+y)}{1+y}dy+\int\frac{\ln(1+y)}{y-1}\right)dy=$$
$$\frac{1}{2}\ln^2(1+y)+\int\left(\frac{\ln 2}{y-1}-\frac{\ln(1-\frac{1-y}{2})}{1-y}\right)dy= \qquad (y=\sqrt{a})$$
$$\frac{1}{2}\ln^2(1+\sqrt{a})+\ln 2\cdot\ln(1-\sqrt{a})-\mathrm{Li}_2(\frac{1-\sqrt{a}}{2})$$

For $a=0$ we have

$$0=-\pi\mathrm{Li}_2(\frac{1}{2})+C=0 \implies C=\frac{\pi^3}{12}-\frac{\pi}{2}\ln^2$$

Now we select the terms diverging for $a\to 1^-$ and obtain

$$-2\pi\ln 2\left(\frac{1}{2}\ln(1-\sqrt{a})+\frac{1}{2(1-\sqrt{a})}\right)-\frac{\pi}{2}\ln(1-\sqrt{a})-\pi\ln 2\cdot\ln(1-\sqrt{a})+$$
$$+2\pi\left(\frac{1}{4}\ln(1-\sqrt{a})+\frac{1}{4}\frac{(1+\sqrt{a})\ln(1+\sqrt{a})}{1-\sqrt{a}}+\ln 2\cdot\ln(1-\sqrt{a})\right)\ \to\ \frac{-\pi}{2}\ln 2-\frac{\pi}{2}$$

The terms not diverging for $a\to 1^-$ contribute

$$-\frac{\pi}{2}\ln 2+\frac{3}{2}\ln^2 2$$

and finally we get

$$-\frac{\pi}{2}\ln 2+\frac{3}{2}\ln^2 2-\frac{\pi}{2}\ln 2-\frac{\pi}{2}+\frac{\pi^3}{12}-\frac{\pi}{2}\ln^2 2=-\pi\ln 2+\pi\ln^2 2-\frac{\pi}{2}+\frac{\pi^3}{12}$$

Comment by Paolo Perfetti: The proposed problem was inspired by num. 11639 of Americam Mathematical Monthly, Aplil–2012 (Omran Kouba). That problem asked to evaluate $\int_0^{\pi/2}(\ln(2\sin x))^2\,dx$.

Solution 2 by Cody Thompson, Allegany College of Maryland, Chaneysville, Pennsylvania, USA and Anastasios Kotronis, Athens, Greece (Jointly). At first, using two basic representations of the Digamma function Ψ, i.e.:

$$\Psi(x)=(\ln\Gamma(x))'=\frac{\Gamma'(x)}{\Gamma(x)} \qquad x\in\mathbb{R}-\mathbb{N} \tag{1}$$

$$\Psi(x)=-\gamma+\sum_{k=1}^{+\infty}\left(\frac{1}{k}-\frac{1}{x+k-1}\right) \qquad x\in\mathbb{R}-\mathbb{N}, \tag{2}$$

we get some special values for Ψ and Ψ'.

From (2) we have that for $x\in\mathbb{R}-\mathbb{N}$:

$$\Psi(x+1)-\Psi(x)=\frac{1}{x}-\lim_{n\to+\infty}\frac{1}{x+n}=\frac{1}{x}. \tag{3}$$

Furthermore, differentiating (4), [1] gives

$$\Psi'(x)=\sum_{k=1}^{+\infty}\frac{1}{(x+k-1)^2} \qquad x>0. \tag{4}$$

On account of the above we have

$$\Psi(1)\overset{(2)}{=}-\gamma$$

$$\Psi(2)\overset{(3)}{=}\Psi(1)+1=1-\gamma$$

$$\Psi\left(\frac{3}{2}\right)\overset{(3)}{=}2+\Psi\left(\frac{1}{2}\right)\overset{(2)}{=}2-\gamma+2\sum_{k=1}^{+\infty}\left(\frac{1}{2k}-\frac{1}{2k-1}\right)=2-\gamma-2\ln 2.$$

$$\Psi'(2)\overset{(4)}{=}\sum_{k=1}^{+\infty}\frac{1}{(k+1)^2}=\frac{\pi^2}{6}-1$$

$$\Psi'\left(\frac{3}{2}\right)\overset{(4)}{=}4\sum_{k=1}^{+\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{2}-4.$$

[1] for $k\geq 1$, $\frac{1}{k}-\frac{1}{x+k-1}$ has a continuous derivative and $\sum_{k=1}^{+\infty}\frac{1}{(x+k-1)^2}$ converges uniformly on $[a,b]$ with $a>0$

Now, for the given integral we have

$$\int_0^{\pi/2} 4\cos^2 x \ln^2(\cos x)\,dx \overset{\cos x=t}{=} 4\int_0^1 \frac{t^2\ln^2 t}{\sqrt{1-t^2}}\,dt.$$

But

$$\int_0^1 \frac{t^{a-1}}{\sqrt{1-t^2}}\,dt \overset{t^2=x}{=} \frac{1}{2}\mathrm{B}\left(\frac{a}{2},\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)}{2\Gamma\left(\frac{a+1}{2}\right)}$$

and by Leibniz's rule, differentiating twice under the integral sign and using (1) we get

$$\int_0^1 \frac{t^{a-1}\ln^2 t}{\sqrt{1-t^2}}\,dt = \frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}\left(\Psi^2\left(\frac{a}{2}\right) - 2\Psi\left(\frac{a+1}{2}\right)\Psi\left(\frac{a}{2}\right) + \Psi^2\left(\frac{a+1}{2}\right) + \Psi'\left(\frac{a}{2}\right) - \Psi'\left(\frac{a+1}{2}\right)\right).$$

For $a=3$ and from the special values of Ψ and Ψ' evaluated above we have that

$$\begin{aligned}\int_0^{\pi/2} 4\cos^2 x\ln^2(\cos x)\,dx &= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\left(\Psi^2\left(\frac{3}{2}\right) - 2\Psi(2)\Psi\left(\frac{3}{2}\right) + \Psi^2(2) + \Psi'\left(\frac{3}{2}\right) - \Psi'(2)\right)\\ &= \frac{\pi^3}{12} + \pi\ln^2 2 - \pi\ln 2 - \frac{\pi}{2}.\end{aligned}$$

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Moubinool Omarjee, Paris, France; and Albert Stadler, Switzerland.

42. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.* Let $f:\mathbb{R}\to\mathbb{R}$ be monotone such that $f+f\circ f\circ f$ is continuous. Prove that f is continuous.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. We argue by contradiction. Let's suppose that f is discontinuous at x_0. The monotonicity (increasing for instance) yields the existence of the right and left limits at any point and in particular at x_0. Thus we have $\lim_{x\to x_0^-} f(x) = l_-$ and $\lim_{x\to x_0^+} f(x) = l_+$ and $l_- \le f(x_0) \le l_+$ but $l_+ - l_- > 0$. Now $f\circ f\circ f$ is also monotone increasing and $L_- = \lim_{x\to x_0^-} f\circ f\circ f \le \lim_{x\to x_0^+} f\circ f\circ f = L_+$. The result is

$$l_- + L_- = \lim_{x\to x_0^-} f + f\circ f\circ f < \lim_{x\to x_0^+} f + f\circ f\circ f = l_+ + L_+$$

Namely, the discontinuity of $f+f\circ f\circ f$ at x_0.

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.
Suppose first that f is nondecreasing. Let x be an arbitrary real number, clearly we have

$$f(x^-) \le f(x) \le f(x^+)$$

Now, if $t < x < u$ then $f(t) \leq f(x) \leq f(u)$, but $f \circ f$ is nondecreasing so

$$f \circ f \circ f(t) \leq f \circ f \circ f(x) \leq f \circ f \circ f(u)$$

or $g(t) - f(t) \leq g(x) - f(x) \leq g(u) - f(u)$. Thus, using the continuity of g at x, and taking the limit as t approaches x by smaller values, and as u approaches x by larger values, we conclude that $g(x) - f(x^-) \leq g(x) - f(x) \leq g(x) - f(x^+)$ or

$$f(x^+) \leq f(x) \leq f(x^-)$$

Combining the above expressions we conclude that $f(x^-) = f(x^+) = f(x)$, so f is continuous at x, which is the desired conclusion since x is arbitrary. A similar argument shows that the conclusion holds if we assume that f is nonincreasing.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania; Albert Standler, Switzerland; and the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

31. Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3, a_2 = 5$ and for all $n \geq 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Prove that

$$1 + 2\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the nth Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

32. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have in common two points X and Y. We draw a line that cuts $\mathcal{C}_1$ at points A and B. Next we draw the lines AX, AY which cut $\mathcal{C}_2$ at points A_X and B_Y and lines BX, BY which cut $\mathcal{C}_2$ at points B_X and B_Y respectively. Show that the three lines AB, A_XB_Y and A_YB_X are parallel.

33. Four dice are thrown at the same time on a table. Find the probability that the sum of the points appeared in the upper faces lies between 14 and 18 points.

34. Let P be an interior point of triangle ABC and let H_A, H_B, H_C be the orthocentres of triangles PBC, PAC, and PAB, respectively. Prove that triangles $H_AH_BH_C$ and ABC have the same area.

35. Compute

$$\lim_{n\to\infty} \frac{1}{n^3} \int_0^n \frac{n^2 + x^2}{5^{-x} + 7}\, dx$$

Solutions

26. *Let $f : [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Show that there exist two unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.*

(22nd Vojtěch Jarník International Mathematical Competition 2012)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. EXISTENCE: Since f is derivable in $[0,1]$, then f is continuous in $[0,1]$. Considering the functions $g(x) = f(x) - x$ and $h(x) = f(x) - (1-x)$ that are continuous in $[0,1]$ and applying Bolzano's theorem we get that exists $\alpha \in [0,1]$ such that $g(\alpha) = 0$ and $\beta \in [0,1]$ with $h(\beta) = 0$. That is, there exist $\alpha, \beta \in [0,1]$ for which $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.
UNIQUENESS: Suppose that exist $\alpha, \alpha' \in [0,1]$, $(\alpha < \alpha')$ such that $f(\alpha) = \alpha$ and $f(\alpha') = \alpha'$. On account of Lagrange's theorem, there exists $\theta \in (\alpha, \alpha') \subset [0,1]$ such that

$$f'(\theta) = \frac{f(\alpha') - f(\alpha)}{\alpha' - \alpha} = \frac{\alpha' - \alpha}{\alpha' - \alpha} = 1$$

contradiction. Likewise, if we assume that there exist $\beta, \beta' \in [0,1]$, $(\beta < \beta')$ such that $f(\beta) = 1 - \beta$ and $f(\beta') = 1 - \beta'$. On account of Lagrange's theorem, there exists $\theta' \in (\beta, \beta') \subset [0,1]$ such that

$$f'(\theta') = \frac{f(\beta') - f(\beta)}{\beta' - \beta} = \frac{(1-\beta') - (1-\beta)}{\beta' - \beta} = -1$$

contradiction. This completes the proof. □

Also solved by Adrian Naco, Tirana, Albania; Henry Ricardo, NY, USA; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

27. *Prove that the equation*

$$(x + y\sqrt{3})^4 + (z + t\sqrt{3})^4 = 7 + 6\sqrt{3}$$

does not have rational solutions.

(Longlist of Catalonian Mathematical Olympiad 2010)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain and Albert Stadler, Herrliberg Switzerland. We give the solution of Díaz-Barrero. First, we observe that for any $a, b, c, d \in \mathbb{Q}$ the expressions

$$(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3} = A + B\sqrt{3}$$

and

$$(a - b\sqrt{3})(c - d\sqrt{3}) = (ac + 3bd) - (ad + bc)\sqrt{3} = A - B\sqrt{3}$$

are equivalents. Applying recursively the preceding, yields

$$(x + y\sqrt{3})^4 = X + Y\sqrt{3} \Leftrightarrow (x - y\sqrt{3})^4 = X - Y\sqrt{3}$$

and

$$(z+t\sqrt{3})^4 = Z+T\sqrt{3} \Leftrightarrow (z-t\sqrt{3})^4 = Z-T\sqrt{3}$$

Therefore,

$$(x+y\sqrt{3})^4+(z+t\sqrt{3})^4 = 7+6\sqrt{3} \Leftrightarrow (x-y\sqrt{3})^4+(z-t\sqrt{3})^4 = 7-6\sqrt{3}$$

The last equality is impossible because $(x-y\sqrt{3})^4+(z-t\sqrt{3})^4 \geq 0$ while $7-6\sqrt{3} < 0$. □

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

28. *Find all polynomials $p(x)$ with real coefficients such that*

$$p(a+b-2c)+p(b+c-2a)+p(c+a-2b) = 3p(a-b)+3p(b-c)+3p(c-a)$$

for all $a,b,c \in \mathbb{R}$.

(Benelux Mathematical Olympiad 2010)

Solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; and José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. We give the solution of Díaz-Barrero. For $a=b=c$, we have $3p(0)=9p(0)$, hence $p(0)=0$. Now set $b=c=0$, then we have

$$p(a)+p(-2a)+p(a) = 3p(a)+3p(-a)$$

for all $a \in \mathbb{R}$. So we get the polynomial equation $p(-2x) = p(x)+3p(-x)$. Note that the zero polynomial is a solution to this equation. Now suppose that p is not the zero polynomial, and let $n \geq 0$ be the degree of p. Let $a_n \neq 0$ be the coefficient of x^n in $p(x)$. Since in the LHS of the polynomial equation the coefficient of x^n is $(-2)^n\, a_n$ while in the RHS the coefficient of x^n is $a_n + 3\,(-1)^n\, a_n$, then $(-2)^n = 1+3\,(-1)^n$. For n even we obtain $2^n = 4$, so, $n=2$, and for n odd, we have $-2^n = -2$, so $n=1$. As we already know that $p(0)=0$, we must have $p(x) = a_2x^2+a_1x$, where a_1, a_2 are real numbers (possibly zero). The polynomial $p(x)=x$ is a solution to our problem because for all $a,b,c \in \mathbb{R}$, we have

$$(a+b-2c)+(b+c-2a)+(c+a-2b) = 3(a-b)+3(b-c)+3(c-a)$$

Likewise, $p(x)=x^2$, is a solution since for all $a,b,c \in \mathbb{R}$ holds

$$\begin{aligned}(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2 &= 6(a^2+b^2+c^2)-6(ab+bc+ca)\\ &= 3(a-b)^2+3(b-c)^2+3(c-a)^2\end{aligned}$$

Now notice that if $p(x)$ and $q(x)$ are solutions to our problems then so is $\lambda\, p(x) + \mu\, q(x)$ for any $\lambda, \mu \in \mathbb{R}$. Therefore, we conclude that for any real numbers a_1, a_2 the polynomial $a_2x^2+a_1x$ is a solution. Furthermore, since we have already shown that there can be no other solutions, these are the only solutions. □

Also solved by Albert Stadler, Herrliberg Switzerland; and José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

29. *Equation $x^3-2x^2-x+1=0$ has three real roots $a>b>c$. Find the value of $ab^2+bc^2+ca^2$.*

Solution 1 by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain. Let us denote by $X = ab^2 + bc^2 + ca^2$ and by $Y = ac^2 + ba^2 + cb^2$. Since $a > b > c$, then $Y - X = (a-c)(c-b)(b-a) > 0$. Therefore, $X < Y$. Let

$$\begin{aligned} u &= a+b+c, \\ v &= ab+bc+ca, \\ w &= abc. \end{aligned}$$

By Cardan-Viète formulae, we have $u = 2, v = -1, w = -1$ and

$$X + Y = a(ab+ac) + b(bc+ba) + c(cb+ca)$$

$$= (a+b+c)(ab+bc+ca) - 3abc = uv - 3w = 1$$

Taking into account the identity

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx),$$

we have

$$XY = (a^3+b^3+c^3-3abc)abc + (a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2) + 9a^2b^2c^2$$

$$= u(u^2-3v)w + v(v^2-3uw) + 9w^2 = -12$$

Taking into account Cardan-Viète formulae again, we have that X and Y are the roots of the equation $t^2 - t - 12 = (t+3)(t-4) = 0$. Since, $X < Y$, then $X = -3$ and $Y = 4$. Therefore, $ab^2 + bc^2 + ca^2 = -3$ and we are done.

□

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Since a, b, c are the roots of $x^3 - 2x^2 - x + 1 = 0$, we conclude that

$$\sigma_1 = a+b+c = 2,\ \sigma_2 = ab+bc+ca = -1,\ \sigma_3 = abc = -1. \tag{1}$$

Let us define u and v by

$$u = ab^2 + bc^2 + ca^2, \qquad v = a^2b + b^2c + c^2a$$

Now, using $a+b+c = 2$ we have

$$\begin{aligned} u+v &= ab(a+b) + bc(b+c) + ca(c+a) \\ &= ab(2-c) + bc(2-a) + ca(2-b) \\ &= 2(ab+bc+ca) - 3abc = -2+3 = 1 \end{aligned}$$

Similarly using $abc = -1$ we have

$$\begin{aligned} uv &= \left(\frac{b}{c} + \frac{c}{a} + \frac{a}{b}\right)\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right) \\ &= 3 + \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} + \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2} \\ &= 3 - a^3 - b^3 - c^3 - \frac{1}{a^3} - \frac{1}{b^3} - \frac{1}{c^3} \end{aligned}$$

Since a, b, c are the roots of $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0$ we see that

$$\begin{aligned} a^3 + b^3 + c^3 &= \sigma_1(a^2 + b^2 + c^2) - (a + b + c)\sigma_2 + 3\sigma_3 \\ &= \sigma_1(\sigma_1^2 - 2\sigma_2) - \sigma_1\sigma_2 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ &= 2^3 + 6 - 3 = 11. \end{aligned}$$

Similarly, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the roots of $x^3 - \frac{\sigma_2}{\sigma_3}x^2 + \frac{\sigma_1}{\sigma_3}x - \frac{1}{\sigma_3} = 0$, so

$$\begin{aligned} \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} &= \frac{\sigma_2^3}{\sigma_3^3} - 3\frac{\sigma_1\sigma_2}{\sigma_3^2} + 3\frac{1}{\sigma_3} \\ &= 1 + 6 - 3 = 4 \end{aligned}$$

Finally, $uv = 3 - 11 - 4 = -12$. Thus, we have proved that $u + v = 1$ and $uv = -12$, so u and v are the roots of the second degree equation $Z^2 - Z - 12 = 0$. On the other hand, we have $u - v = (a - b)(b - c)(c - a) < 0$, that is $u < v$. This allows us to identify u as the smallest root of the second degree equation $Z^2 - Z - 12 = 0$. That is $u = -3$.

□

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.

30. *Let $\{a_n\}_{n\geq 1}$ be a strictly increasing sequence of positive integers such that $\lim\limits_{n\to+\infty} \dfrac{a_{n+1}}{a_1 a_2 \ldots a_n} = +\infty$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ is an irrational number.*

(Training Sessions of Spanish Math Team for VJIMC 2012)

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. From the statement immediately follows that for any $c > 1$ there exists n_c such that for all $n > n_c$ holds

$$a_{n+1} > c\,(a_1 \cdot a_2 \cdots a_n),$$

and then

$$a_{n+2} > c\,(a_1 \cdots a_n a_{n+1}) > c^2\,(a_1 \cdots a_n)^2 > c^2\,(a_1 \cdots a_n, \quad \ldots \quad a_{n+k}) > c^k\,(a_1 \cdots a_n)$$

Now suppose that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{p}{q}$ is a rational number. Then we would have

$$q(a_1 \cdot a_2 \cdots a_n) \sum_{k=1}^{n} \frac{1}{a_k} + p(a_1 \cdot a_2 \cdots a_n) \sum_{k=n+1}^{\infty} \frac{1}{a_k} = p(a_1 \cdot a_2 \cdots a_n)$$

so the following number

$$p\,(a_1 \cdot a_2 \cdots a_n) \sum_{k=n+1}^{\infty} \frac{1}{a_k}$$

must be an integer for any positive integer n. But if $c > p + 1 > 1$ then holds

$$0 < p(a_1 \cdot a_2 \cdots a_n) \sum_{k=n+1}^{\infty} \frac{1}{a_k} \leq p \sum_{k=n+1}^{\infty} \frac{1}{c^{k-n}} = \frac{p}{c-1} < 1$$

contradicting what it was assumed and the proof is complete.

□

Solution 2 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Since $\lim_{n\to+\infty} \frac{a_{n+1}}{a_1a_2\dots a_n} = +\infty$, then there exists a positive integer n_0 such that for all $n \geq n_0$, is $\frac{a_{n+1}}{a_1a_2\dots a_n} \geq 2$. So, $a_{n+1} \geq 2(a_1a_2\dots a_n) \geq 2^n$, for all $n \geq n_0$. Therefore, $\frac{1}{a_{n+1}} \leq \frac{1}{2^n}$ for all $n \geq n_0$ and the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. To see that its sum is an irrational number we argue by contradiction. Assume that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{p}{q}$, where p and q are integers. Since $\lim_{n\to+\infty} \frac{a_{n+1}}{a_1a_2\dots a_n} = +\infty$, then there exists $n_1 \geq 2$ such that $\frac{a_{n+1}}{q(a_1a_2\dots a_n)} \geq 3$, for all $n \geq n_1$. From the preceding, for all $n \geq n_1$ holds

$$a_{n+1} \geq 3q(a_1a_2\dots a_n) \geq 3 \tag{5}$$

On the other hand,

$$p(a_1a_2\dots a_{n_1}) = \sum_{k=1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k} = \sum_{k=1}^{n_1} \frac{a_1a_2\dots a_{n_1}q}{a_k} + \sum_{k=n_1+1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k}$$

Since $p(a_1a_2\dots a_{n_1})$ and $\sum_{k=n_1+1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k}$ are both integers, then $\sum_{k=n_1+1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k}$ is also an integer. From (5) immediately follows that for all $k > n_1 + 1$, holds

$$0 < \frac{(a_1a_2\dots a_{n_1})q}{a_k} = \frac{(a_1a_2\dots a_{n_1}a_{n_1+1}a_{n_1+2}\dots a_{k-1})q}{a_{n_1+1}a_{n_1+2}\dots a_{k-1}a_k} \leq \left(\frac{1}{3}\right)^{k-n_1}$$

Therefore,

$$0 < \sum_{k=n_1+1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k} \leq \sum_{k=n_1+1}^{\infty} \left(\frac{1}{3}\right)^{k-n_1} = \frac{1}{2}$$

which is not possible because $\sum_{k=n_1+1}^{\infty} \frac{a_1a_2\dots a_{n_1}q}{a_k}$ is an integer. This completes the proof.

□

Solution 3 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_1a_2\dots a_n} = +\infty$, there is some $n_0 \geq 3$ such that $a_{n+1} \geq a_1a_2\dots a_n$ for every $n \geq n_0$. Since clearly $a_2 \geq 2$ we conclude, in particular, that $a_{n+1} \geq 2a_n$ for every $n \geq n_0$. This proves the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$. Now, let α denote the sum of the series, and let

$$S_n = \sum_{k=1}^{n} \frac{1}{a_k}, \qquad R_n = \alpha - S_n = \sum_{k=1}^{\infty} \frac{1}{a_{n+k}}$$

For $n \geq n_0$, and $k \geq 1$, we have

$$\frac{a_{n+1}}{a_{n+k}} = \prod_{j=1}^{k-1} \frac{a_{n+j}}{a_{n+j+1}} \leq \prod_{j=1}^{k-1} \frac{1}{2} = \frac{1}{2^{k-1}}.$$

Thus, for $n \geq n_0$ we have

$$R_n = \frac{1}{a_{n+1}} \sum_{k=1}^{\infty} \frac{a_{n+1}}{a_{n+k}} \leq \frac{1}{a_{n+1}} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \leq \frac{2}{a_{n+1}}.$$

Now, suppose that α is a rational number, that is $\alpha = p/q$ where p and q are positive integers. Because $\lim_{n\to\infty} \frac{a_{n+1}}{a_1 a_2 \ldots a_n} = +\infty$, there is an integer $m > n_0$ such that $\frac{a_{m+1}}{a_1 a_2 \ldots a_m} > 2q$. So,

$$0 < R_m \leq \frac{2}{a_{m+1}} < \frac{1}{q a_1 a_2 \ldots a_m}$$

but this is equivalent to

$$0 < \frac{p}{q} - S_m < \frac{1}{q a_1 a_2 \ldots a_m}$$

or

$$0 < p(a_1 a_2 \ldots a_m) - q(a_1 a_2 \ldots a_m) S_m < 1$$

which is absurd since the number $p(a_1 a_2 \ldots a_m) - q(a_1 a_2 \ldots a_m) S_m$ is clearly an integer. This contradiction proves that α is an irrational.

□

Solution 4 by Henry Ricardo, NY, USA. Suppose that $\sum_{n=1}^{\infty} 1/a_n = p/q$, where p and q are integers such that $(p, q) = 1$. The hypothesis implies the existence of a positive integer N such that $a_{N+1}/a_1 a_2 \ldots a_N q > 2$ and $a_{n+1} > 2a_n$ for $n \geq N + 1$. Then

$$p a_1 a_2 \ldots a_N = \sum_{n=1}^{\infty} \frac{a_1 a_2 \ldots a_N q}{a_n} = \sum_{n=1}^{N} \frac{a_1 a_2 \ldots a_N q}{a_n} + \sum_{n=N+1}^{\infty} \frac{a_1 a_2 \ldots a_N q}{a_n}$$

We note that $p a_1 a_2 \ldots a_N$ is an integer, as it the finite sum on the right. However, because of the way we chose N, the series $\sum_{n=N+1}^{\infty} \frac{a_1 a_2 \ldots a_N q}{a_n}$ is term-by-term less than the series $1/2 + 1/4 + \ldots + 1/2^n + \ldots$, whose sum is 1. This contradiction establishes the desired result.

□

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

Editors Comment. Paolo Perfetti acknowledged us that this problem was proposed by Paul Erdös and appeared with solution as Problem 4321, *American Mathematical Monthly*, Vol.57, No.5 (1950) 347–348.

MATHNOTES SECTION

An Optimal Inequality of the Tangent Function

Omran Kouba

Abstract. In this note we deal with some inequalities for the tangent function that are valid for all x in $(-\pi/2, \pi/2)$. These inequalities are optimal in the sense that the best values of the exponents involved are obtained.

1. Introduction

The starting point was when I wanted to provide my students of "Basic Calculus" class, with a way to prove that

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$$

without using any advanced topics like the L'Hôpital's rule. So, I came up with the following

Proposition 1. *For every $x \in (0, \pi/2)$ holds*

$$x + \frac{x^3}{3} < \tan x < x + \frac{\tan^3 x}{3}$$

Clearly, the above limit follows immediately from the Proposition 1. But this was not the end of the story, it was just the beginning of my investigation. In fact, the preceding inequality states that $3(\tan x - x)$ is somewhere between x^3 and $\tan^3 x$, but where exactly?

In order to describe our results, an important role is played by the family of functions $(f_\gamma)_{\gamma\in[0,3]}$ defined on $[0, \pi/2)$ by

$$f_\gamma(x) = x^{3-\gamma} \tan^\gamma x$$

Taking into account the well-known inequality $\tan x \geq x$ for $0 \leq x < \pi/2$, we see that the family $(f_\gamma)_{\gamma\in[0,3]}$ is increasing in the sense that $f_\alpha \leq f_\beta$ for $\alpha < \beta$. Using this family, we can reformulate the inequality writing it in the form

$$f_0(x) < 3(\tan x - x) < f_3(x), \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

So, it is natural to be interested in identifying the best α and β such that $f_\alpha(x) < 3(\tan x - x) < f_\beta(x)$ for $0 < x < \pi/2$. We were able to completely answer this question, our results are summarized in the following two statement

Proposition 2. *If for every $x \in (0, \pi/2)$ we have $f_\alpha(x) \leq 3(\tan x - x) \leq f_\beta(x)$, where f_γ was defined previously, then $\alpha \leq 1$ and $\beta \geq 6/5$.*

Theorem 1. *The following two inequalities hold*

(*a*) *For every* $x \in (0, \pi/2)$ *we have* $f_1(x) < 3(\tan x - x)$,

(*b*) *For every* $x \in (0, \pi/2)$ *we have* $3(\tan x - x) < f_{6/5}(x)$.

Equivalently, for all $x \in \left(0, \frac{\pi}{2}\right)$ *holds*

$$x + \frac{1}{3}x^2 \tan x < \tan x < x + \frac{1}{3}x^{9/5} \tan^{6/5} x \tag{6}$$

Before proving our results, it is worth mentioning that there is a lot of similar inequalities involving trigonometric functions in the literature ([1, 2, 3, 4]). For instance, the Becker-Stark's inequality [1] states

$$\frac{8x}{\pi^2 - 4x^2} < \tan x < \frac{\pi^2 x}{\pi^2 - 4x^2}, \qquad \text{for } 0 < x < \tfrac{\pi}{2}$$

Also, in [4] the authors prove, among other things, that for $0 < x < \frac{\pi}{2}$, one has

$$x + \frac{x^3}{3} + \frac{2}{15}x^4 \tan x < \tan x < x + \frac{x^3}{3} + \left(\frac{2}{\pi}\right)^4 x^4 \tan x \tag{7}$$

Numerical evidence shows that the upper bound in (6) is sharper than the upper bound in (7) for $x \in (0, x_0)$ where $x_0 \approx 1.2332$, and that the lower bound in (6) is sharper than the lower bound in (6) for $x \in (x_1, \pi/2)$ where $x_1 \approx 1.5255$. So the two results are complementary but not comparable.

2. **Results and Proofs**

Clearly, the next Proposition 3 follows from our main Theorem 1, but it can be proved directly.

Proposition 3. *For every* $x \in (0, \pi/2)$ *the following inequality holds:*

$$x + \frac{x^3}{3} < \tan x < x + \frac{\tan^3 x}{3}$$

Proof. Indeed, let g and h be the functions defined on $[0, \pi/2)$ by

$$g(x) = \tan x - x - \frac{x^3}{3},$$
$$h(x) = \frac{\tan^3 x}{3} + x - \tan x.$$

Clearly, for $x \in (0, \pi/2)$, we have $g'(x) = \tan^2 x - x^2 > 0$ and $h'(x) = \tan^4 x > 0$. Thus, both g and h are increasing on the interval $(0, \pi/2)$, and the desired inequality follows since $g(0) = h(0) = 0$. □

Proposition 4. *If for some* $0 \le \alpha, \beta \le 3$, *we have*

$$\left(\frac{\tan x}{x}\right)^\alpha \le \frac{3(\tan x - x)}{x^3} \le \left(\frac{\tan x}{x}\right)^\beta$$

for every $x \in (0, \pi/2)$, *then* $\alpha \le 1$ *and* $\beta \ge 6/5$.

Proof. Suppose that for $x \in (0, \pi/2)$ we have

$$\left(\frac{\tan x}{x}\right)^{\alpha} \leq \frac{3(\tan x - x)}{x^3} \leq \left(\frac{\tan x}{x}\right)^{\beta},$$

that is $\alpha \leq \phi(x) \leq \beta$ where ϕ is defined on $(0, \pi/2)$ by

$$\phi(x) = \log\left(\frac{3(\tan x - x)}{x^3}\right) \Big/ \log\left(\frac{\tan x}{x}\right).$$

Now, since

$$\phi(x) = \frac{\log(\tan x) + \log(1 - x/\tan x) + \log 3 - 3\log(x)}{\log(\tan x) - \log x}$$

we conclude that

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \phi(x) = 1$$

On the other hand, since in the neighborhood of 0 we have

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2}{15}x^4 + O(x^6),$$

we get

$$\begin{aligned} \log\left(\frac{\tan x}{x}\right) &= \frac{x^2}{3} + O(x^4) \\ \log\left(\frac{3(\tan x - x)}{x^3}\right) &= \log\left(1 + \frac{2}{5}x^2 + O(x^4)\right) \\ &= \frac{2}{5}x^2 + O(x^4) \end{aligned}$$

Thus, $\phi(x) = \frac{6}{5} + O(x^2)$, and consequently

$$\lim_{x \to 0^+} \phi(x) = \frac{6}{5}.$$

Therefore, he preceding and the fact that $\alpha \leq \phi(x) \leq \beta$ for every $x \in (0, \pi/2)$, imply that $\alpha \leq 1$ and $\beta \geq \frac{6}{5}$ as desired. □

To prove our main theorem, we need the following technical lemma.

Lema 1. *Let ϕ be the function defined on $\mathbb{R}$ by*

$$\phi(x) = (9 - 24x^2)\cos(x) - 9\cos(3x) - 4x\sin(3x)$$

Then $\phi(x) > 0$ for $0 < x \leq \pi/2$.

Proof. In order to determine the sign of $\phi(x)$ for $x \in (0, \pi/2]$, we will use power series expansion. Clearly, for every real x we have

$$\begin{aligned} \phi(x) &= (9 - 24x^2)\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} - 9\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!} - 4x\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^{2n-1} x^{2n-1}}{(2n-1)!} \\ &= \sum_{n=0}^{\infty} \left(9 + 24(2n)(2n-1) - 9 \cdot 3^{2n} + 4(2n) \cdot 3^{2n-1}\right) \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 3\sum_{n=0}^{\infty} \left(32n^2 - 16n + 3) + (8n - 27) \cdot 3^{2n-2}\right) \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

Thus, for a real x we have

$$\phi(x) = 3\sum_{n=0}^{\infty}(-1)^n \frac{T_n}{(2n)!}x^{2n},$$

where,

$$T_n = 2(4n-1)^2 + 1 + (8n-27)9^{n-1}.$$

Noting that $T_0 = T_1 = T_2 = T_3 = 0$ we conclude that

$$\phi(x) = 3\sum_{n=4}^{\infty}(-1)^n \frac{T_n}{(2n)!}x^{2n}$$

Since $T_n > 0$ for $n \geq 4$, then the preceding series is alternate. Now, if we prove that the sequence $\left(\frac{T_n}{(2n)!}x^{2n}\right)_{n\geq 4}$ is decreasing for any $x \in (0, \pi/2]$ then this would imply that $\phi(x) > 0$ for $x \in (0, \pi/2]$, because the first term in the last series is positive.

Let U_n be defined by,

$$U_n = (2n+2)(2n+1)T_n - 3T_{n+1}.$$

a simple computation shows that

$$\begin{aligned} U_n &= (4n^2+6n+2)\big(32n^2 - 16n + 3 + (8n-27)9^{n-1}\big) \\ &\qquad - 3(32n^2 + 48n + 19 + (8n-19)9^n) \\ &= 128n^4 + 128n^3 - 116n^2 - 158n - 51 + (32n^3 - 60n^2 - 362n + 459)9^{n-1} \\ &= B_n + A_n \cdot 9^{n-1}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} B_n &= 128n^4 + 128n^3 - 116n^2 - 158n - 51 \\ A_n &= 32n^3 - 60n^2 - 362n + 459 \end{aligned}$$

Now, it is straightforward to check that

$$\begin{aligned} B_{n+1} &= 128n^4 + 640n^3 + 1036n^2 + 437n + 69(n-1), \\ A_{n+4} &= 32n^3 + 324n^2 + 694n + 99. \end{aligned}$$

Thus, A_n and B_n are positive for $n \geq 4$ on account that $U_n > 0$ for $n \geq 4$. Using U_n again, we conclude that for $n \geq 4$ and $x \in (0, \sqrt{3}]$

$$(2n+2)(2n+1)T_n > x^2 T_{n+1}$$

or, equivalently,

$$\forall\, n \geq 4, \quad \forall\, x \in \left(0, \sqrt{3}\right], \quad \frac{T_n}{(2n)!}x^{2n} > \frac{T_{n+1}}{(2n+2)!}x^{2n+2}.$$

It follows that the sequence $\left(\frac{T_n}{(2n)!}x^{2n}\right)_{n\geq 4}$ is decreasing for any $x \in (0, \sqrt{3}]$, and, as we have already explained, this implies that $\phi(x) > 0$ for $x \in \left(0, \sqrt{3}\right]$, and Lemma 1 follows since $\frac{\pi}{2} < \sqrt{3}$. $\square$

Now, we are ready to prove our Main result.

Proof of Theorem 1. (a) Consider the function g defined on the interval $(0, \pi/2)$ by

$$g(x) = 3 - x^2 - 3x \cot x$$

We have

$$g'(x) = x - 3\cot x + 3x\cot^2 x = (1 + 3\cot^2 x)h(x),$$

where $h(x) = x - \dfrac{3\tan x}{3 + \tan^2 x}$. Similarly h has a derivative on $[0, \pi/2)$ given by

$$\begin{aligned} h'(x) &= 1 - 3\frac{(3 - \tan^2 x)(1 + \tan^2 x)}{(3 + \tan^2 x)^2} \\ &= \frac{4\tan^2 x}{(3 + \tan^2 x)^2} \end{aligned}$$

So, h is increasing, with $h(0) = 0$. This implies that h is positive on the interval $(0, \pi/2)$. Going back we conclude that g is also increasing on $(0, \pi/2)$. Finally, since $\lim_{x\to 0^+} g(x) = 0$, we conclude that g is positive on $(0, \pi/2)$, but it is straightforward to check that this is equivalent to the fact that $3(\tan x - x) > x^2 \tan x$ for $x \in (0, \pi/2)$ which is the desired inequality.

(b) Again, we will consider an auxiliary function. Let g be the function defined on $(0, \pi/2)$ by

$$g(x) = 6\log\left(\frac{\tan x}{x}\right) - 5\log\left(\frac{3(\tan x - x)}{x^3}\right).$$

We have

$$\begin{aligned} g'(x) &= \frac{6}{\cos x\, \sin x} + \frac{9}{x} - \frac{5\sin^2 x}{\cos x(\sin x - x\cos x)} \\ &= \frac{(9 - 6x^2)\cos x + x(4\sin^3 x - 3\sin x) - 9\cos^3 x}{x\cos x\, \sin x\,(\sin x - x\cos x)} \end{aligned}$$

So, recalling the expression of $\cos(3x)$ and $\sin(3x)$ in terms of $\cos x$ and $\sin x$ we see that

$$\begin{aligned} g'(x) &= \frac{(9 - 24x^2)\cos(x) - 9\cos(3x) - 4x\sin(3x)}{4x\cos^2 x\, \sin x\,(\tan x - x)}, \\ &= \frac{\phi(x)}{4x\cos^2 x\, \sin x\,(\tan x - x)}, \end{aligned} \tag{9}$$

where ϕ is the function considered in Lemma 1. Using this Lemma we see that g is increasing on $(0, \pi/2)$. But $\lim_{x\to 0^+} g(x) = 0$, so g is positive on $(0, \pi/2)$, and this is equivalent to $3(\tan x - x) < x^{9/5}\tan^{6/5} x$ which is the desired inequality.

□

Corollary 1. *The necessary and sufficient condition, on the real numbers α and β, for the following inequality*

$$1 + \frac{x^2}{3}\left(\frac{\tan x}{x}\right)^{\alpha} < \frac{\tan x}{x} < 1 + \frac{x^2}{3}\left(\frac{\tan x}{x}\right)^{\beta}$$

to hold for every nonzero real x from $(-\pi/2, \pi/2)$, is that $\alpha \le 1$ and $\beta \ge 6/5$.

Proof. This follows from Proposition 4, Theorem 1, and from the fact that the functions considered are even. □

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